COV886 Special Module in Algorithms: Computational Social Choice

## Lecture 9

# Cake Cutting

March 12, 2022 | Rohit Vaish

Reminder about starting recording

# Fair Division

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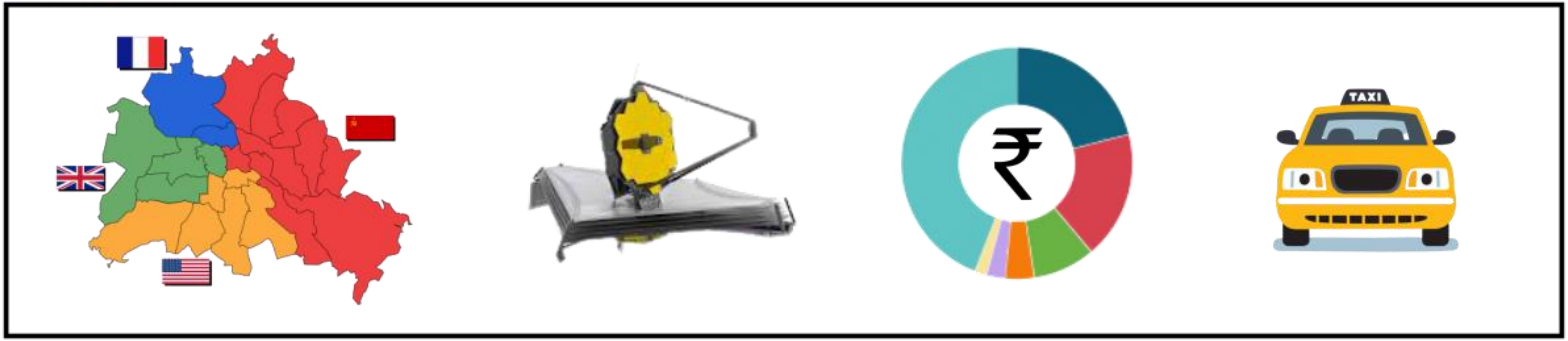


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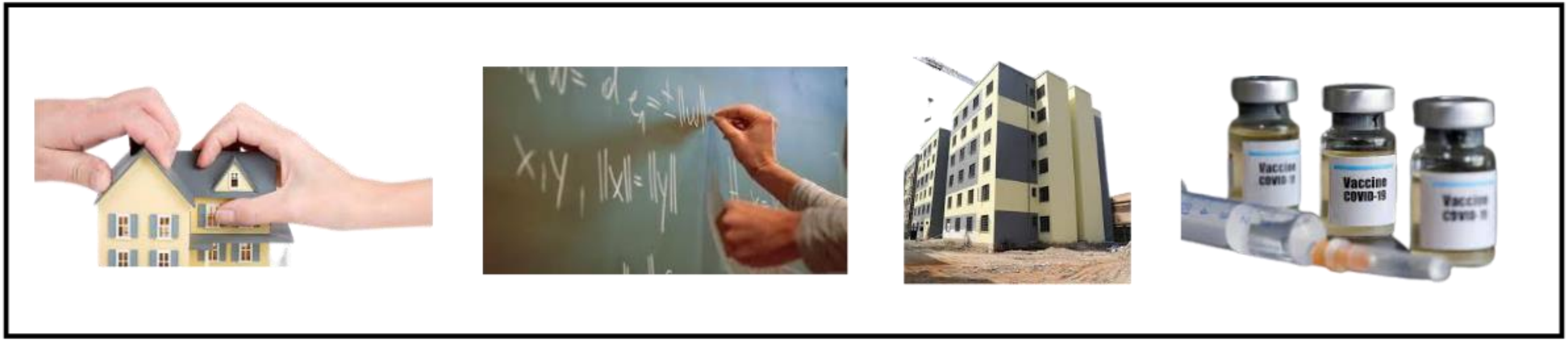

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agents need not be identical

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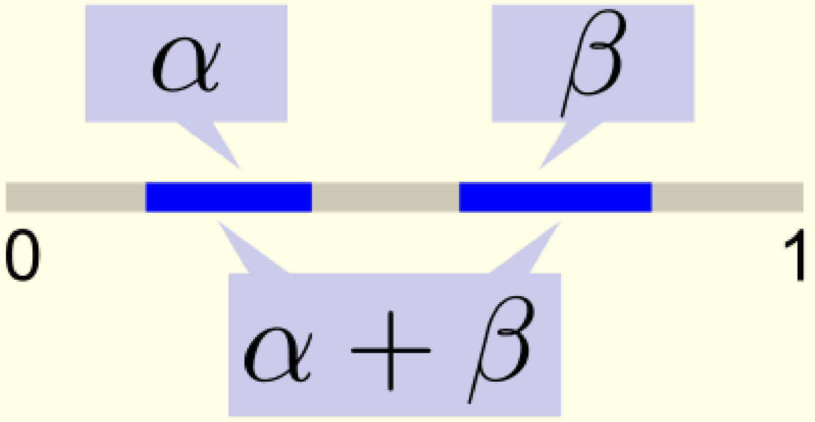

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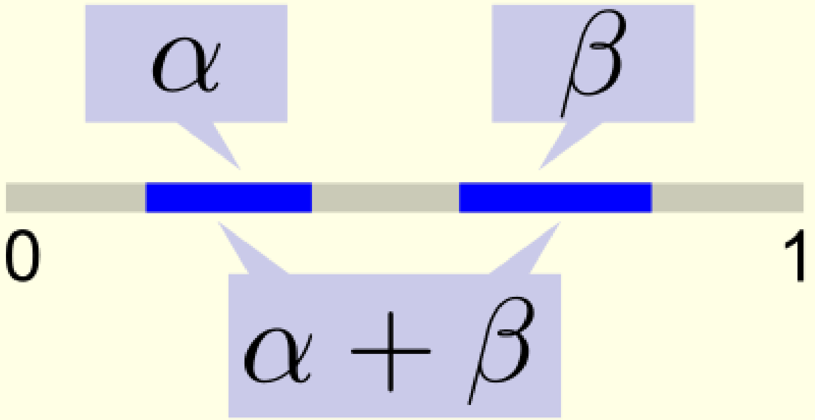


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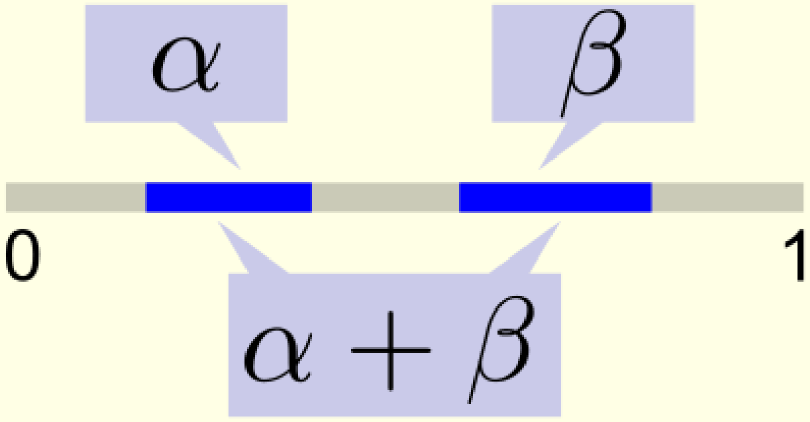


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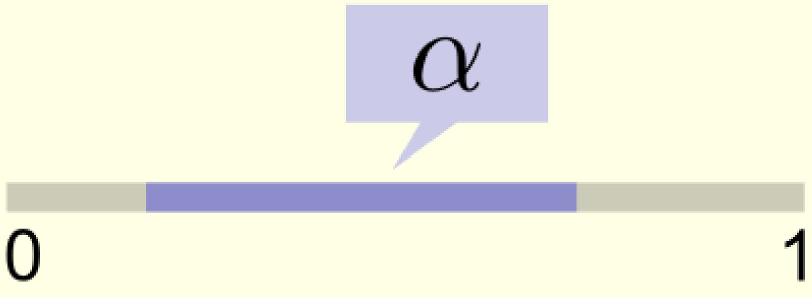

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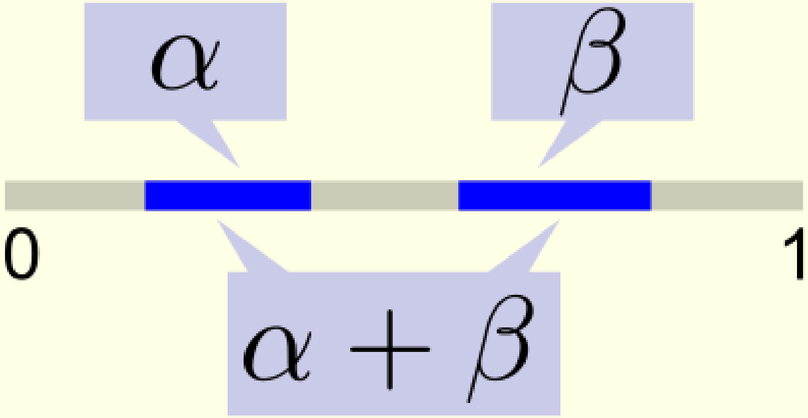


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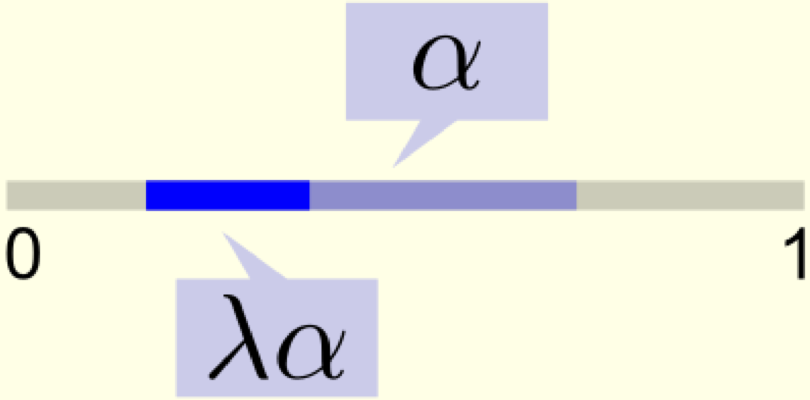

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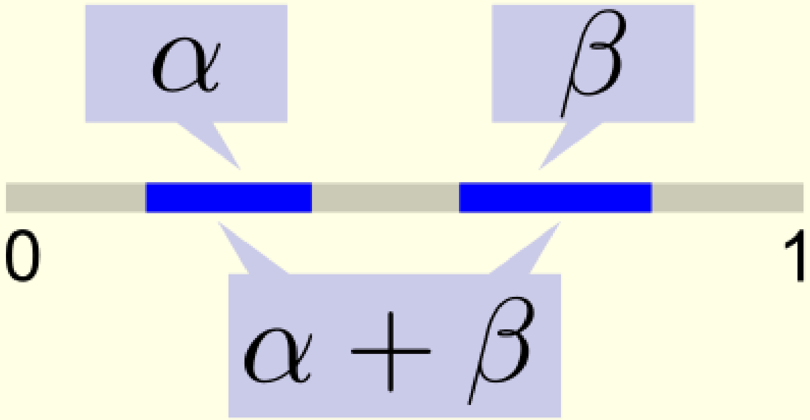


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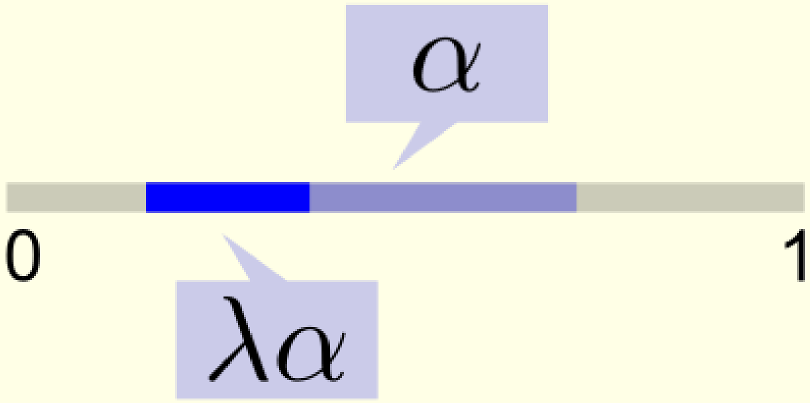


Normalization: for each agent $i$, $v_i([0,1]) = 1$.

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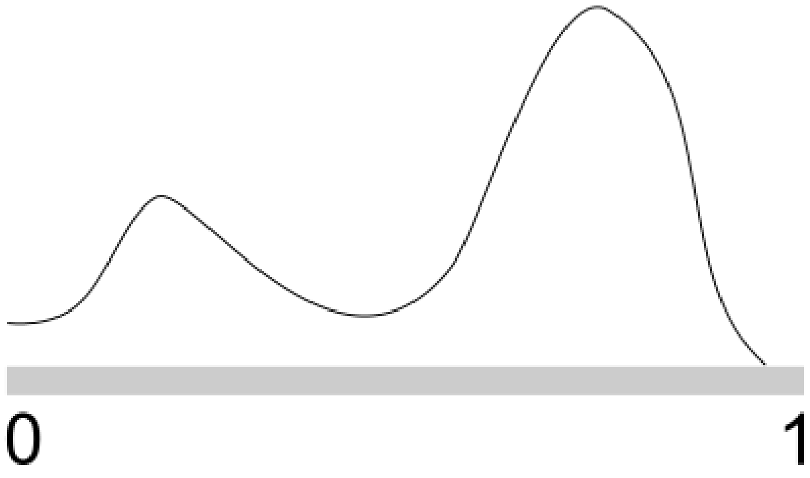

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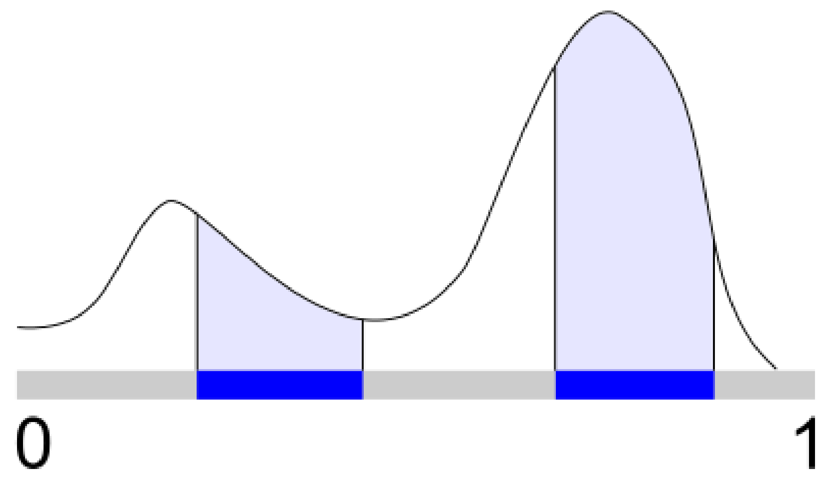

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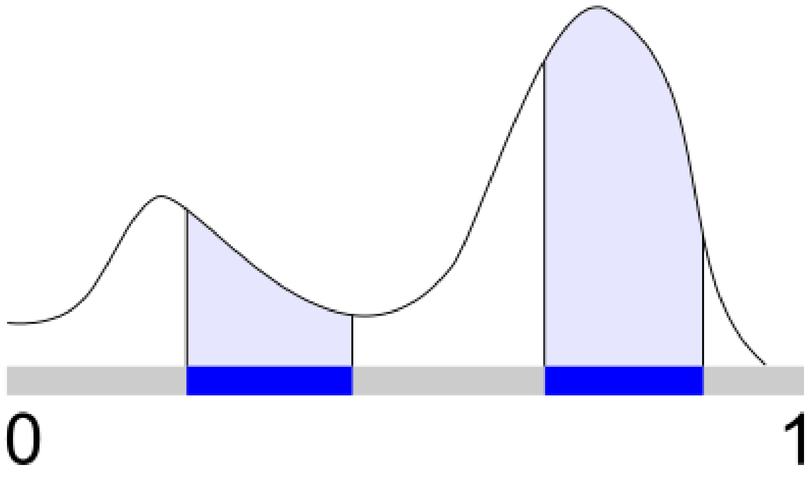
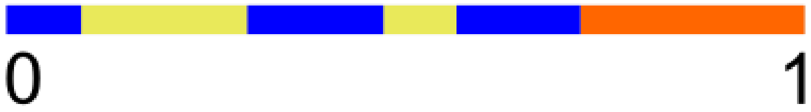
*value density function*



# Fairness notions

- *Allocation/Division*: A partition $(A_1, A_2, \ldots, A_n)$ of the cake [0,1] where each $A_i$ is a piece of cake.



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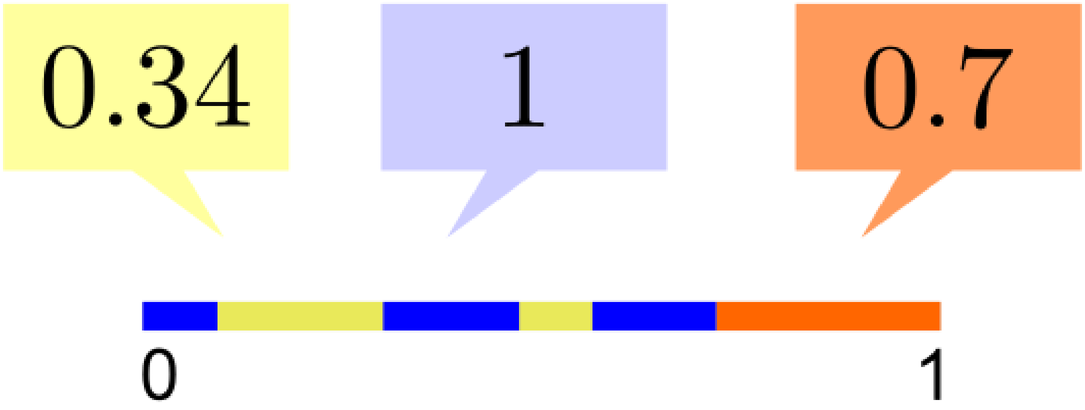

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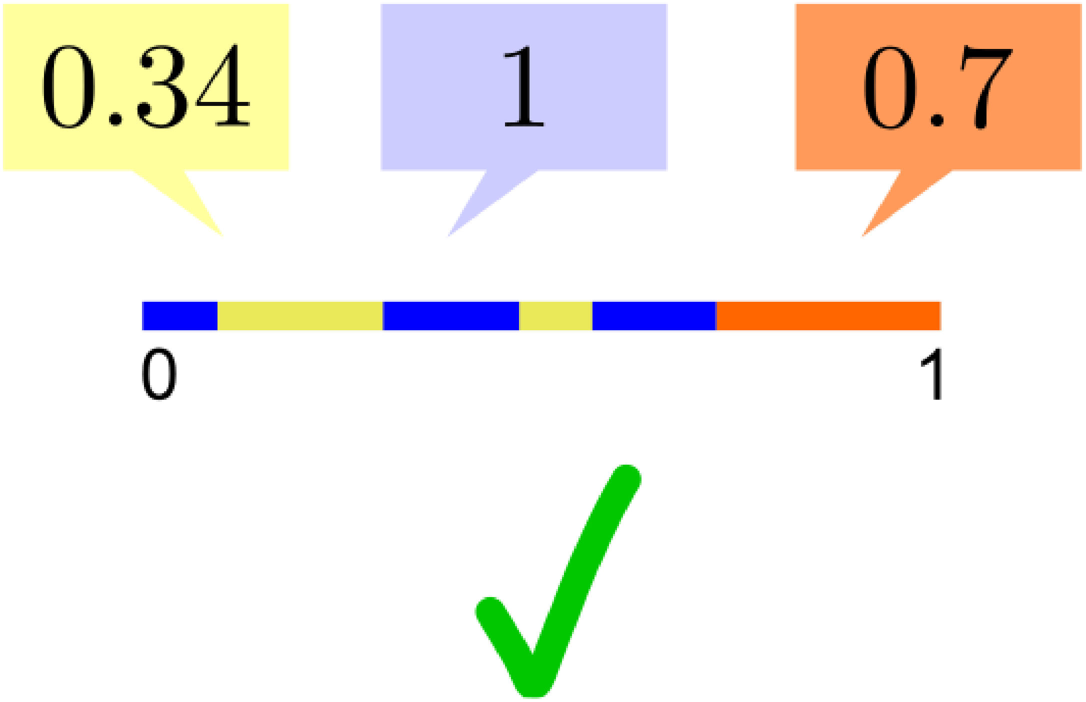

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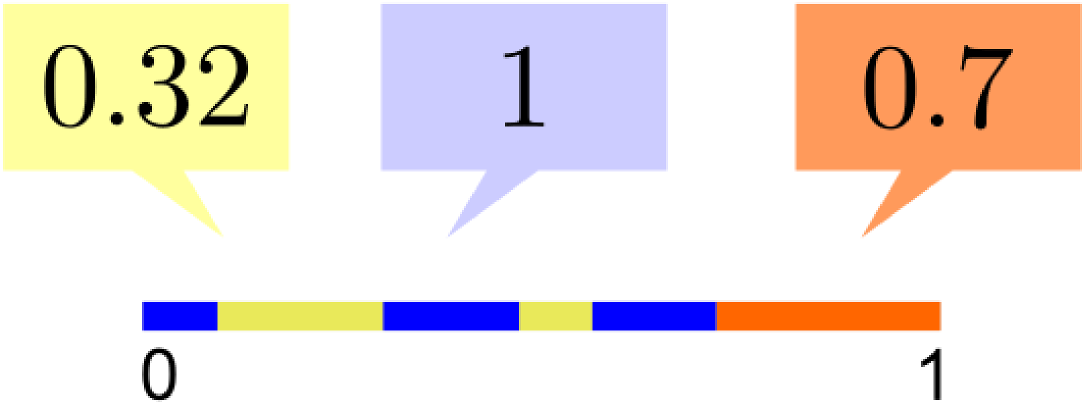

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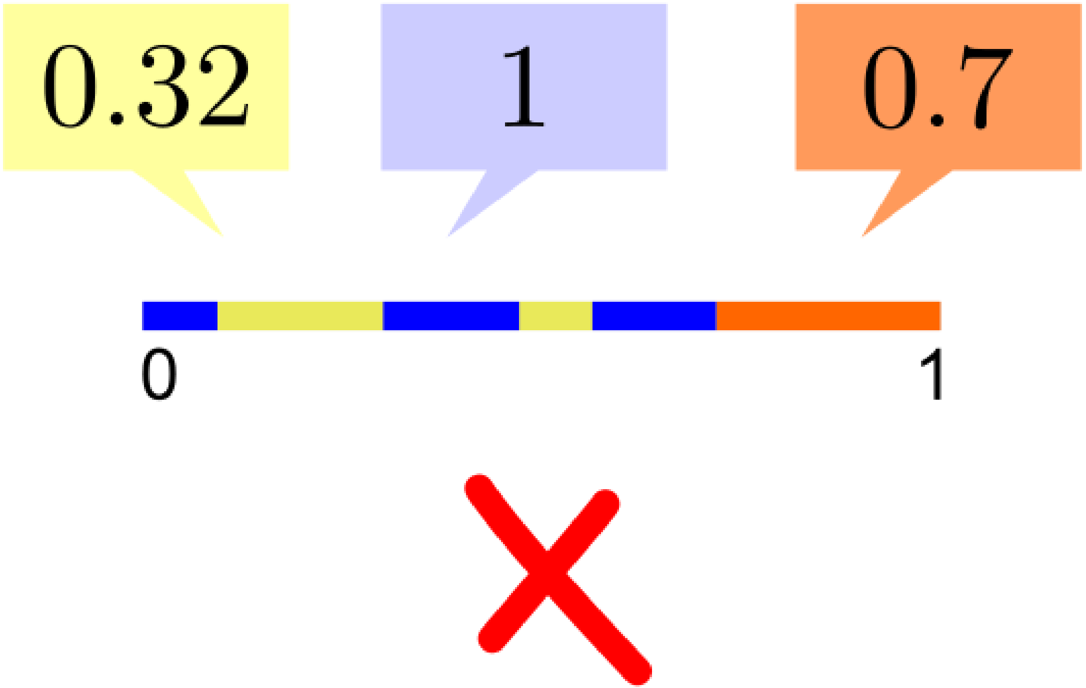

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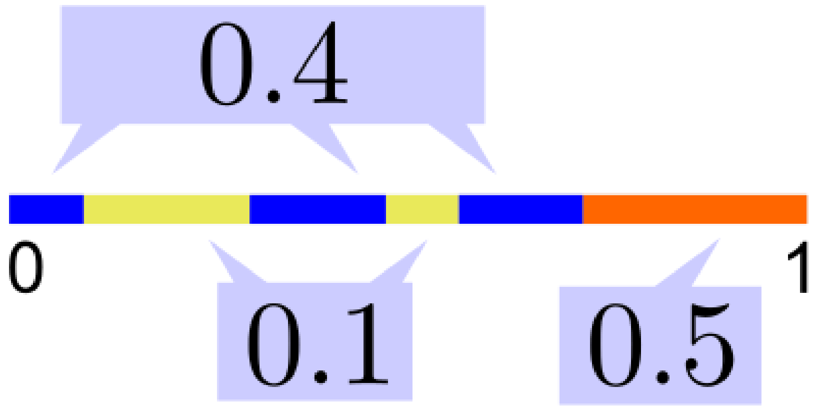


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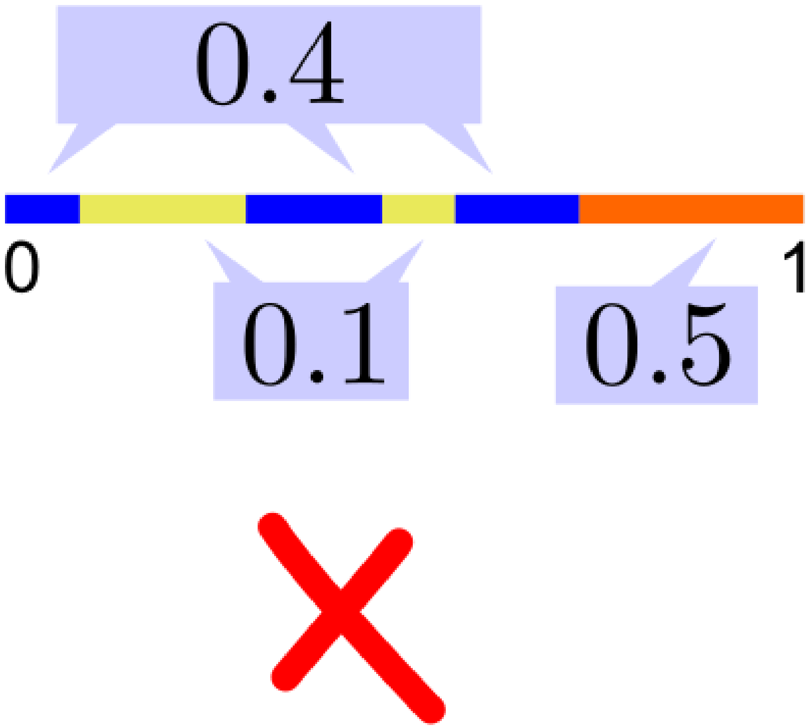


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For two agents (n=2), is one property stronger than the other?

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What about three or more agents?

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Prop implies EF for *two* agents (but no more)

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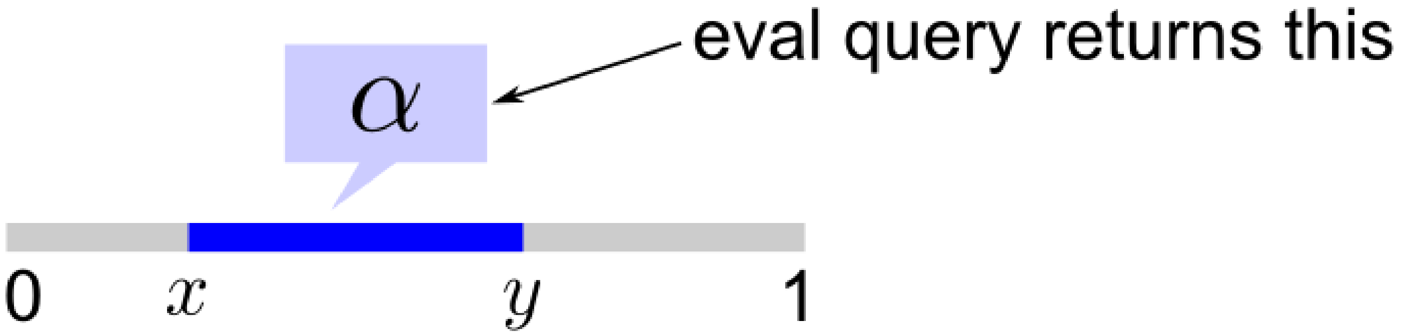

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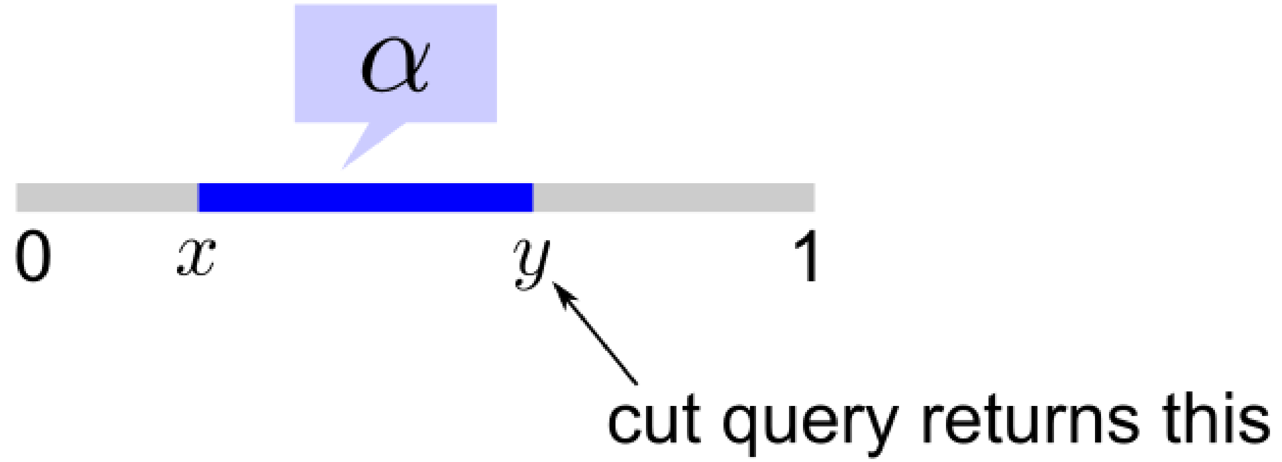

# Cake-Cutting Algorithms

Let's start by thinking about proportionality for two agents.

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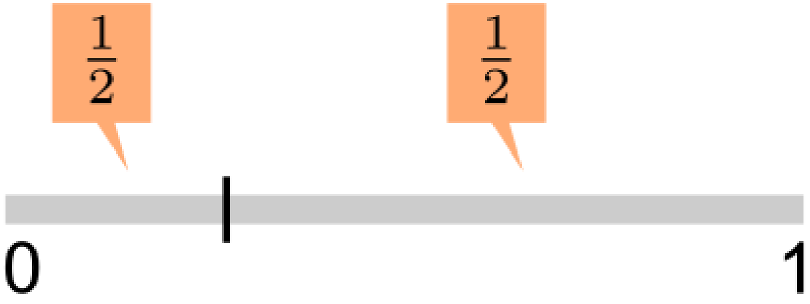

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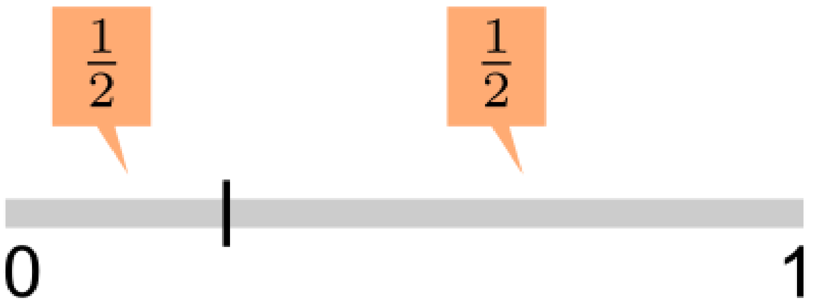

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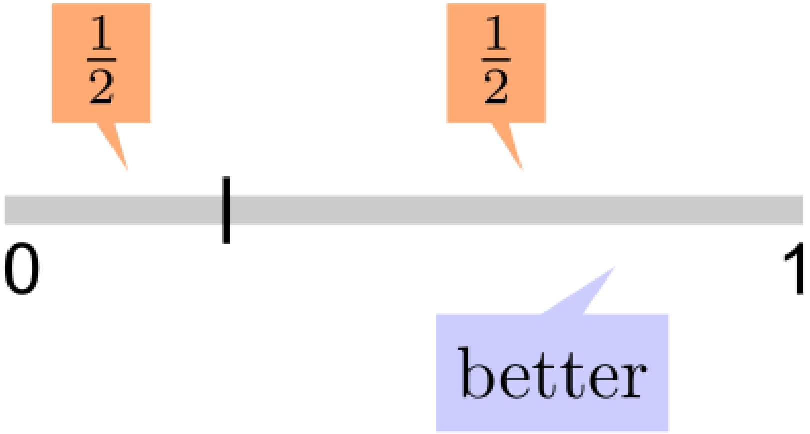

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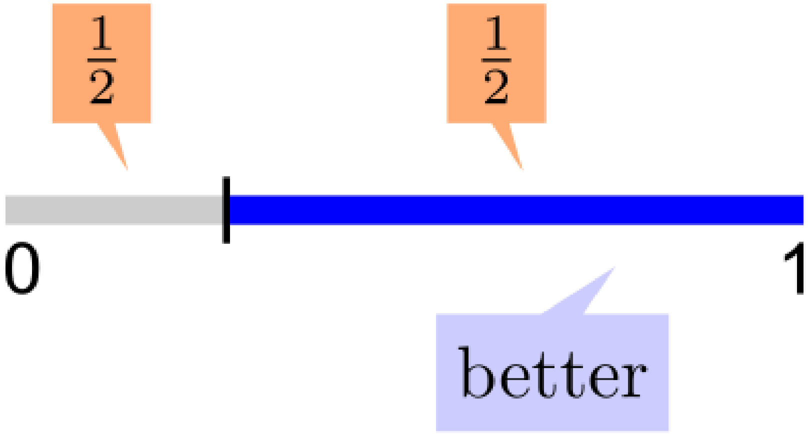

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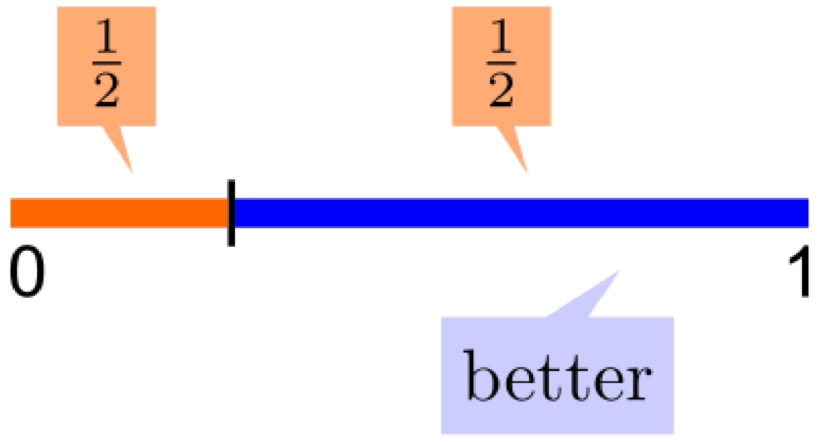

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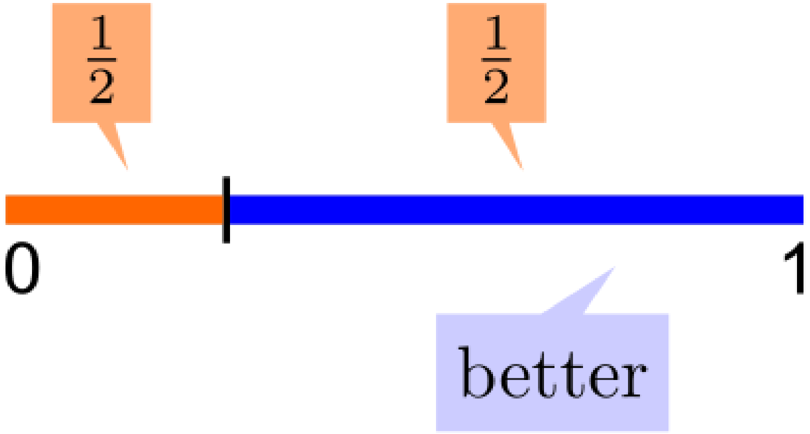


Is the cut-and-choose outcome proportional?

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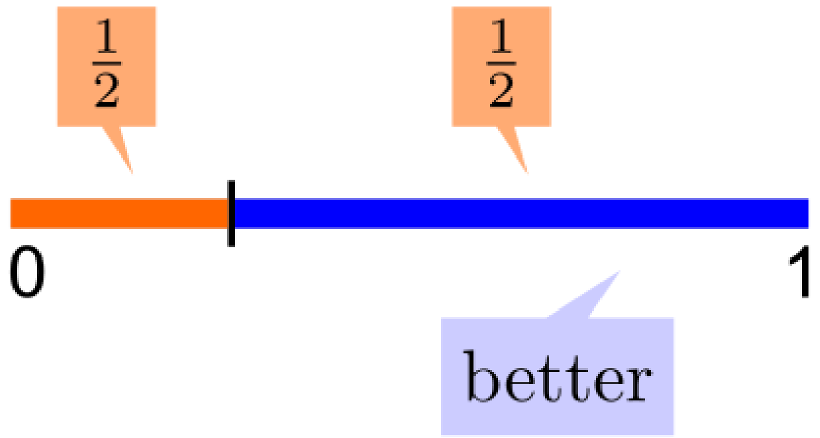


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Yes! Agent 2's value is at least 1/2. Agent 1's value is exactly 1/2.

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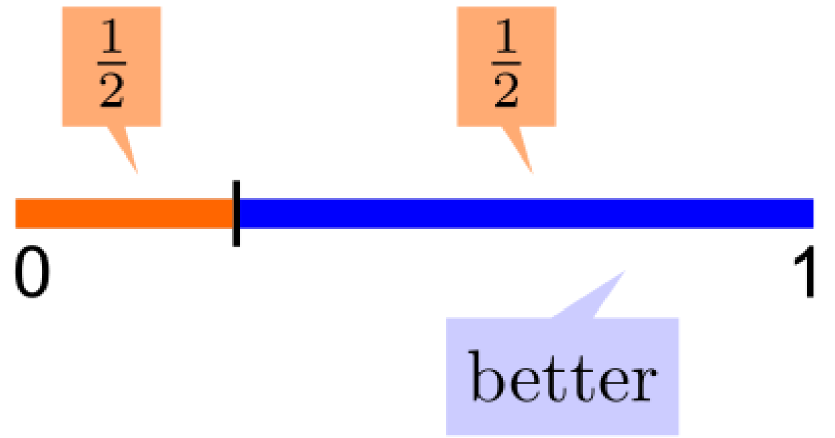


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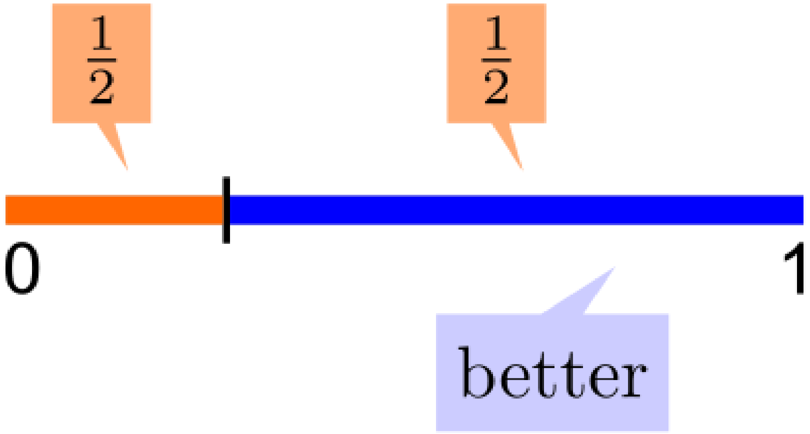


Is the cut-and-choose outcome envy-free?

Yes! EF and Prop are equivalent for two agents.

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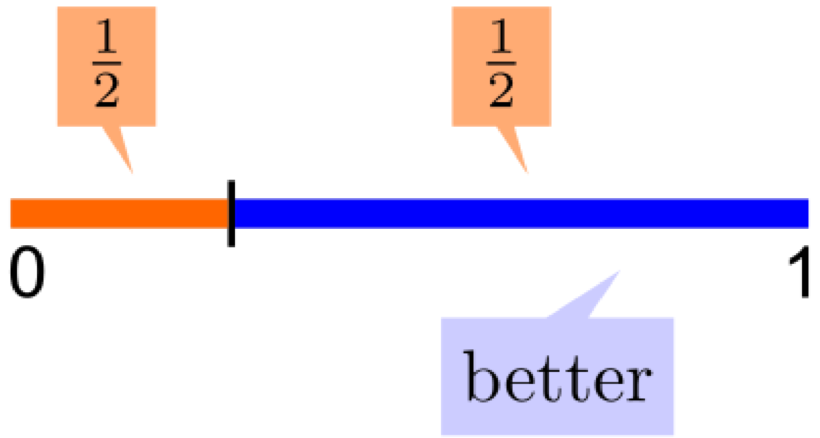


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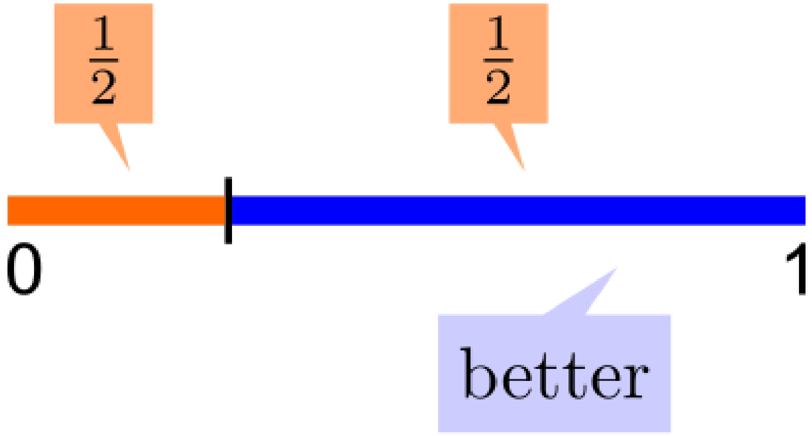


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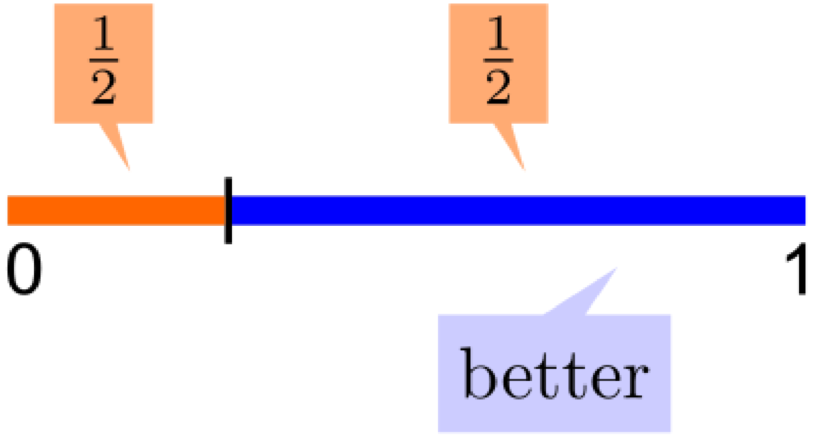


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$$\texttt{eval}_2(0, y)$$

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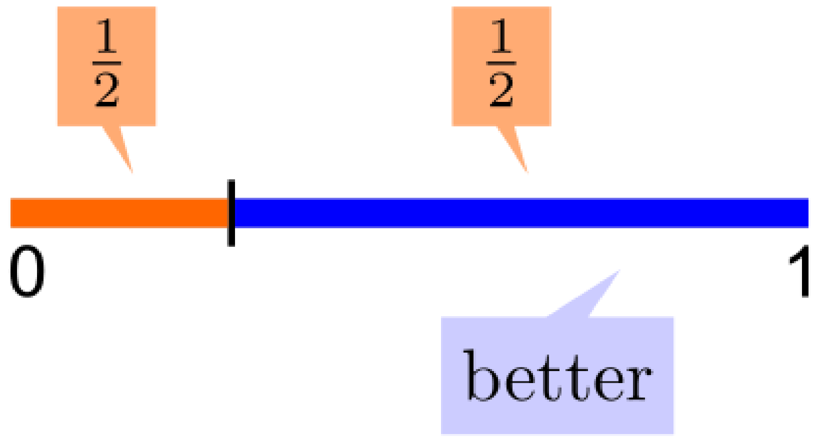


For two agents, an envy-free/proportional cake division can be computed using two queries.

# Dubins-Spanier Procedure

A proportional cake division protocol for any number of agents

# Dubins-Spanier Procedure

1. A referee gradually moves a knife from left to right.

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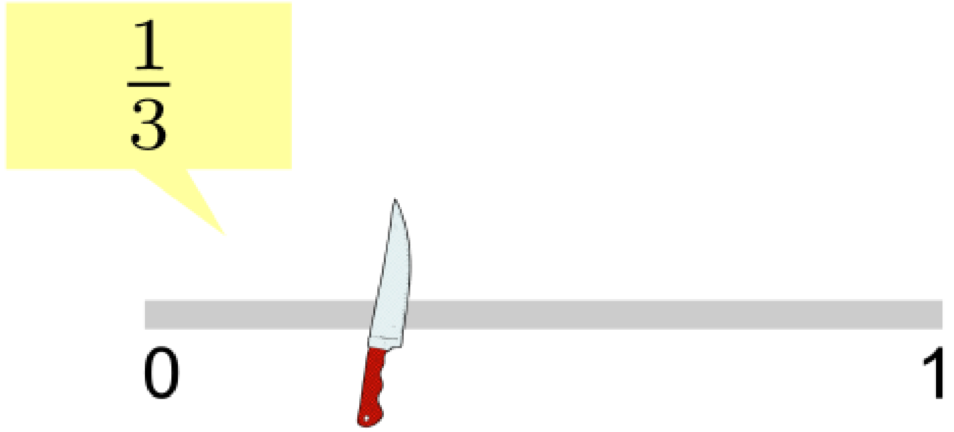

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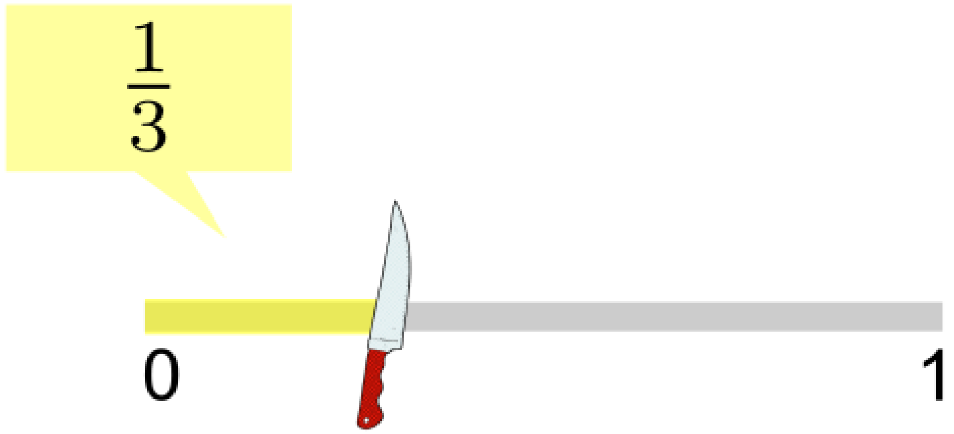

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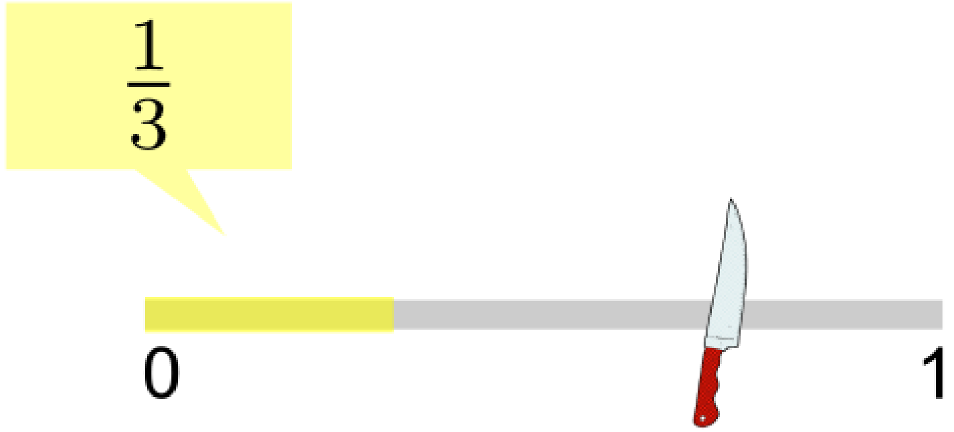

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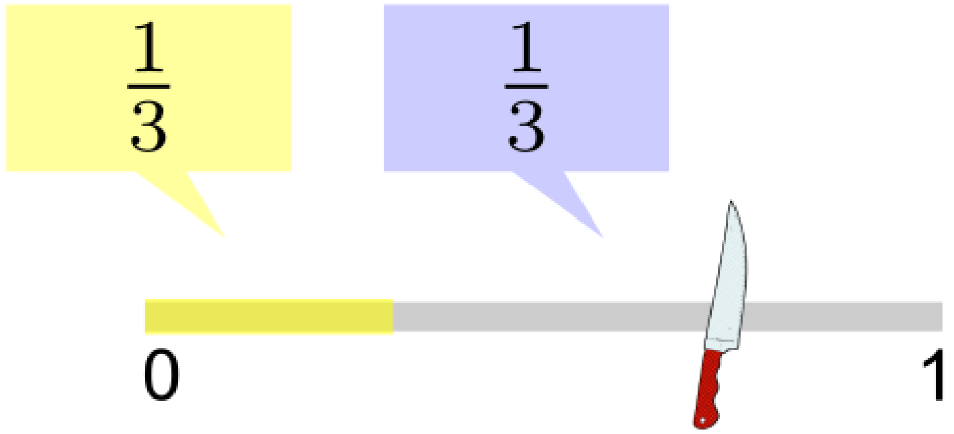

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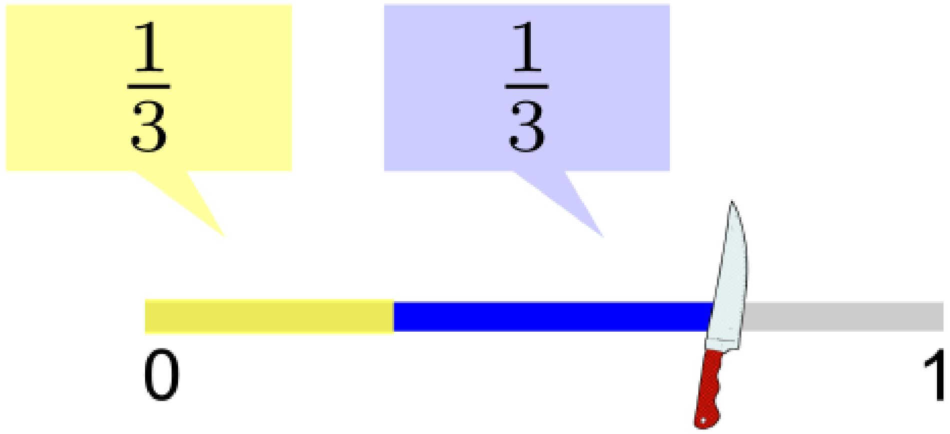

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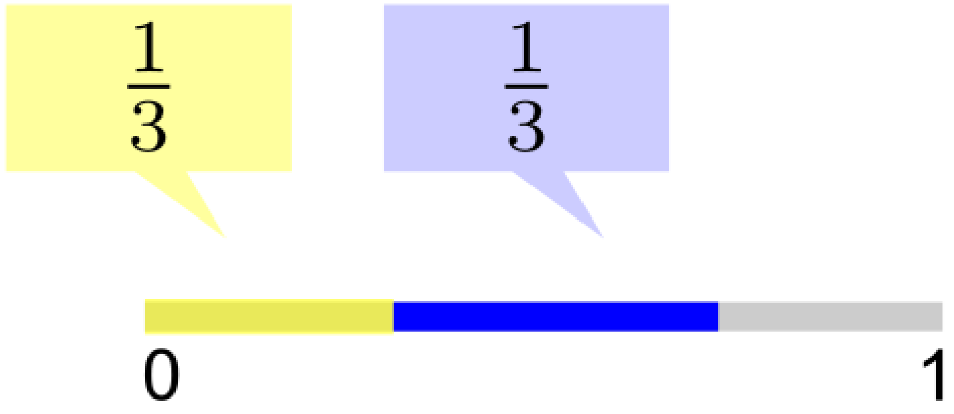

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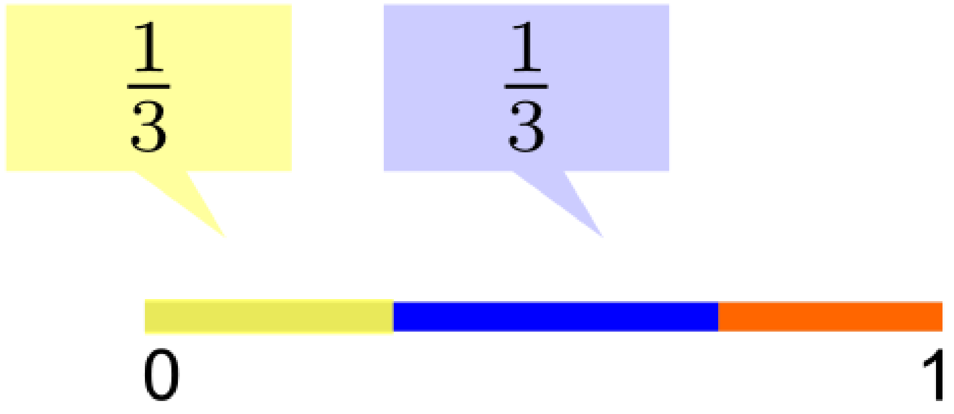

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Why is the resulting allocation proportional?

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Why is the resulting allocation proportional?

Every agent except for the last one gets *exactly* 1/n.
The last agent gets *at least* 1/n.

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Can this procedure be implemented in the Robertson-Webb model?

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Can this procedure be implemented in the Robertson-Webb model?

Yes!

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Query complexity in the Robertson-Webb model?

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Query complexity in the Robertson-Webb model?

$\mathcal{O}(n^2)$ queries (Exercise)

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For $n$ agents, a proportional cake division can be computed using $O(n^2)$ queries.

# The Story of Proportionality

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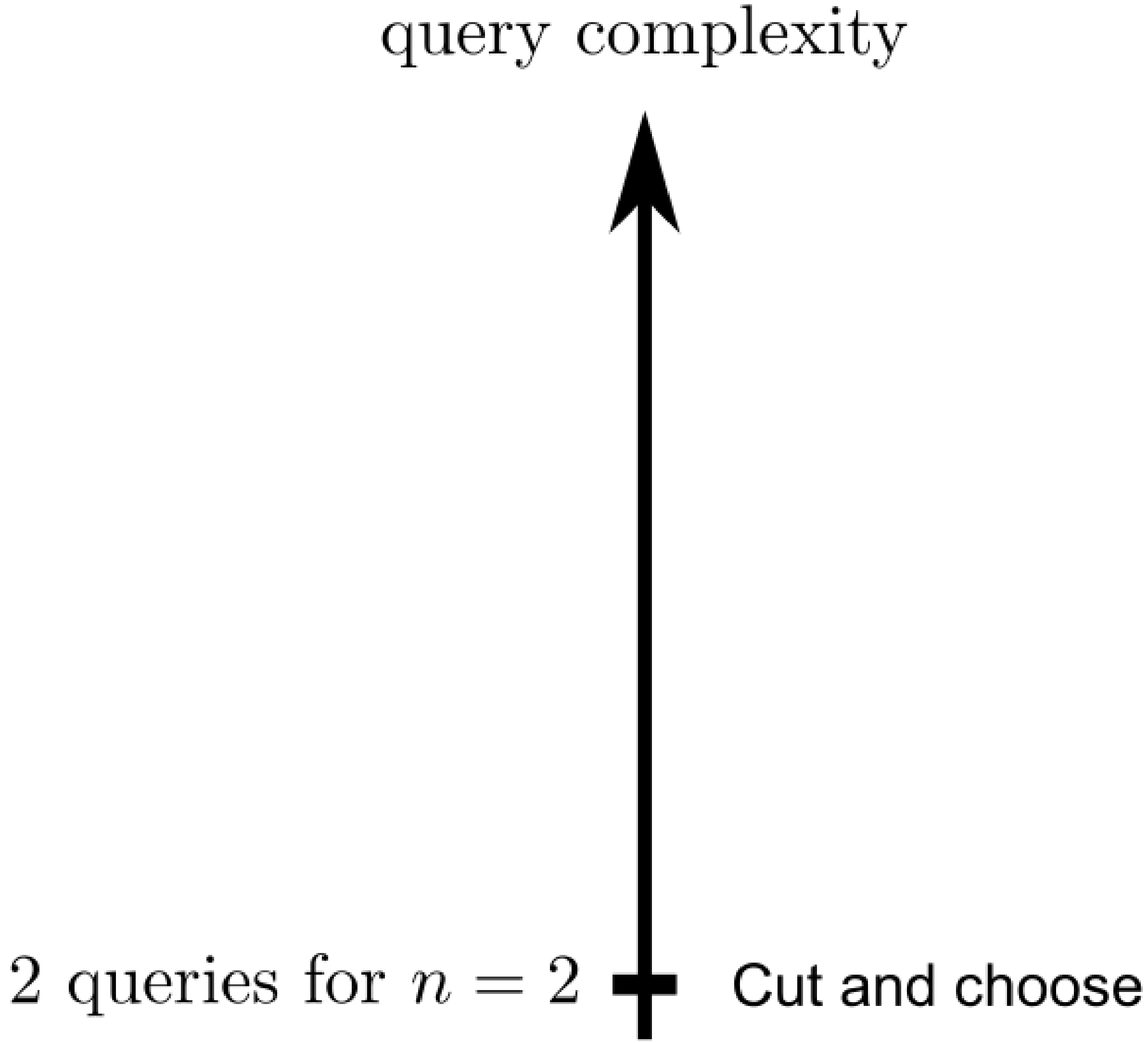
query complexity

# The Story of Proportionality



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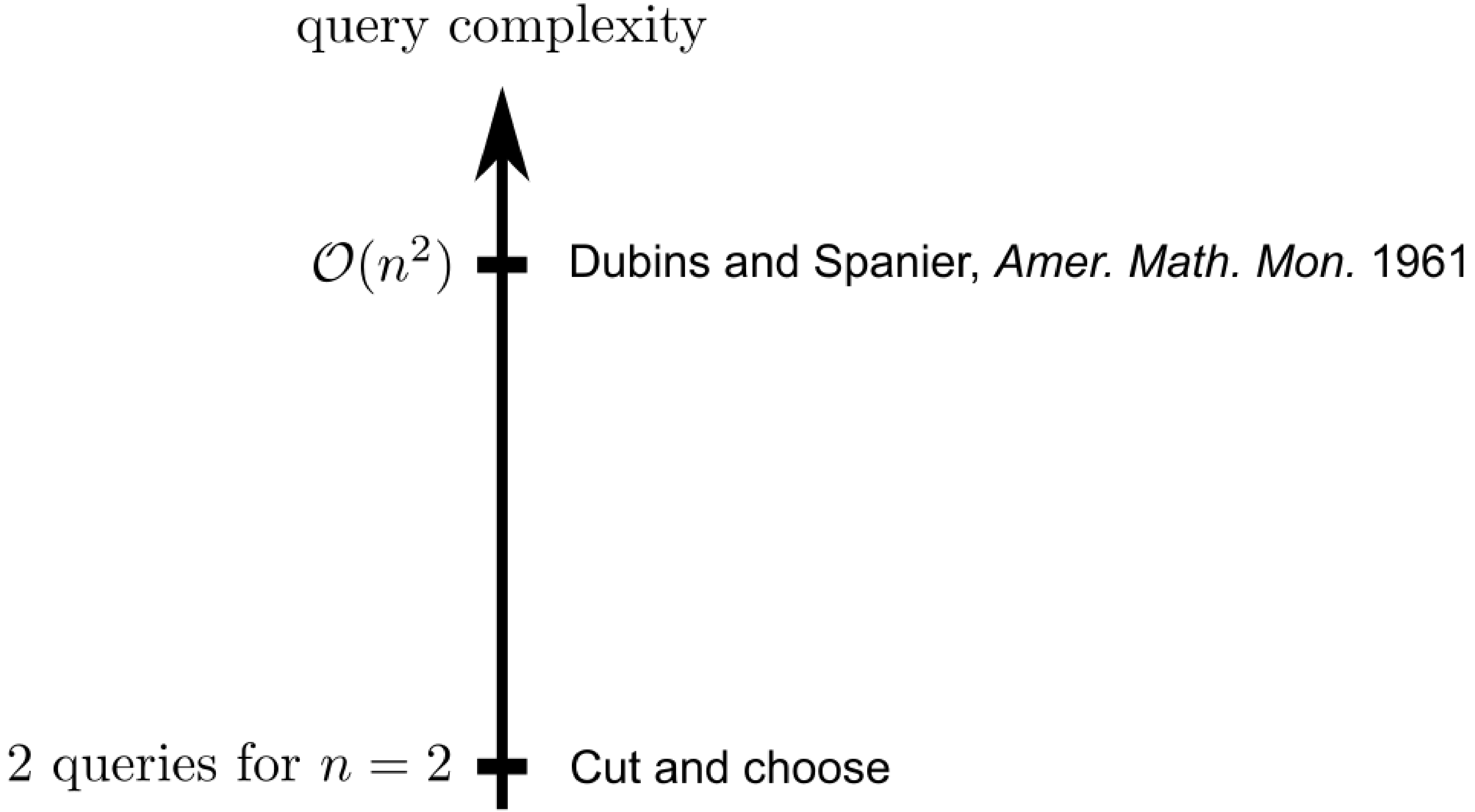

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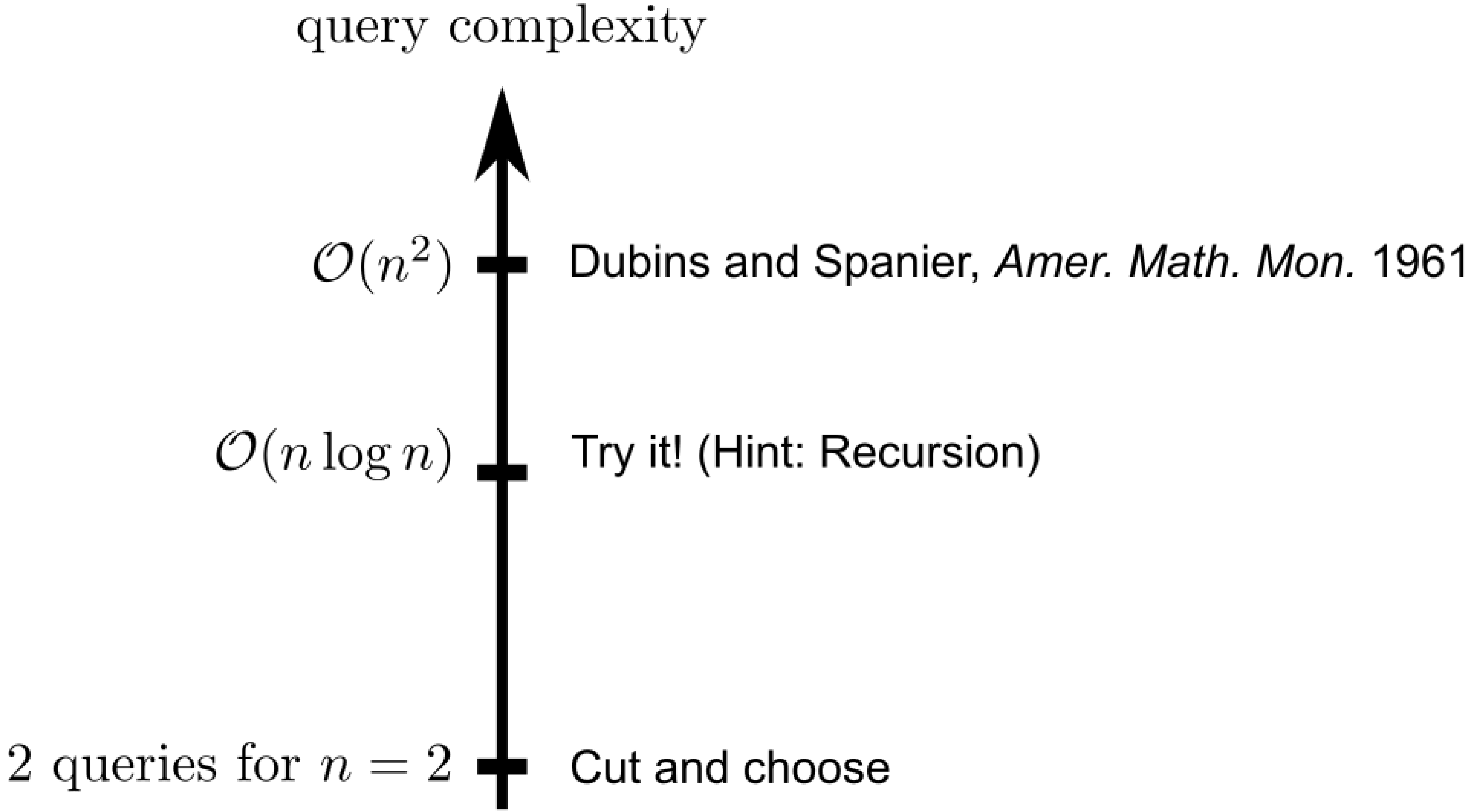

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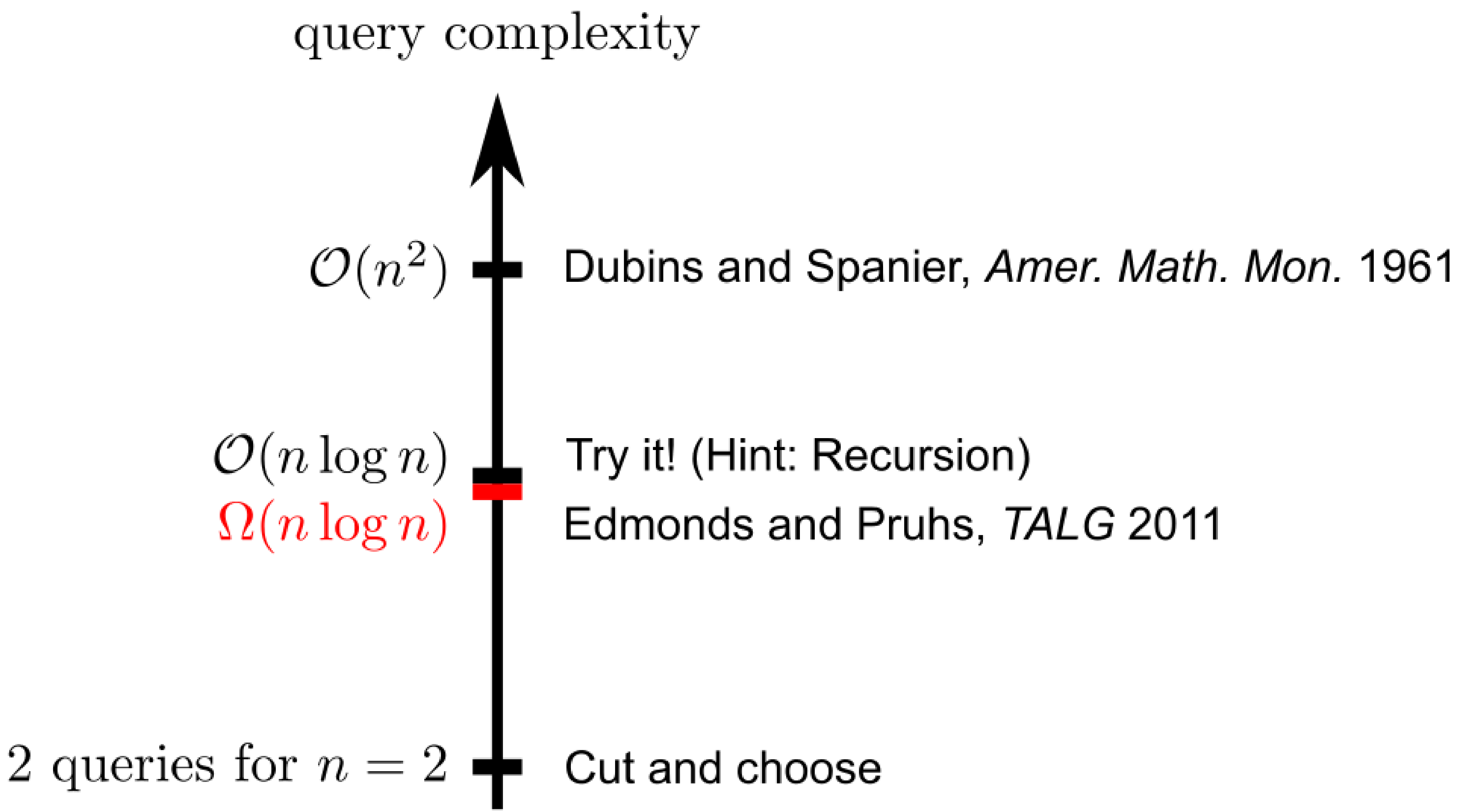

# The Story of Envy-freeness

# Selfridge-Conway Procedure

An envy-free cake division protocol for three agents

# Selfridge-Conway Procedure

## Phase 1

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1. Agent A divides the cake into three equal pieces (as per $v_A$).

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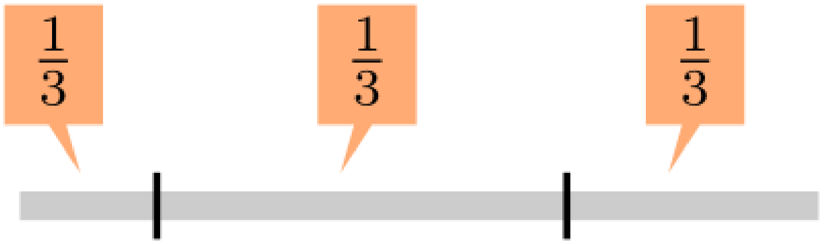

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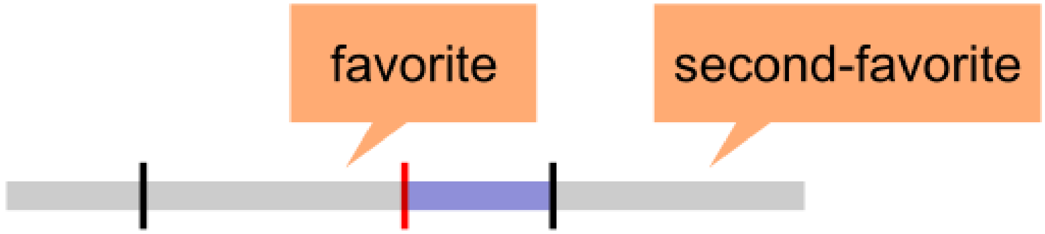

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Let T = owner of the trimmed piece (T = B or C); let T' = B∪C \ T.

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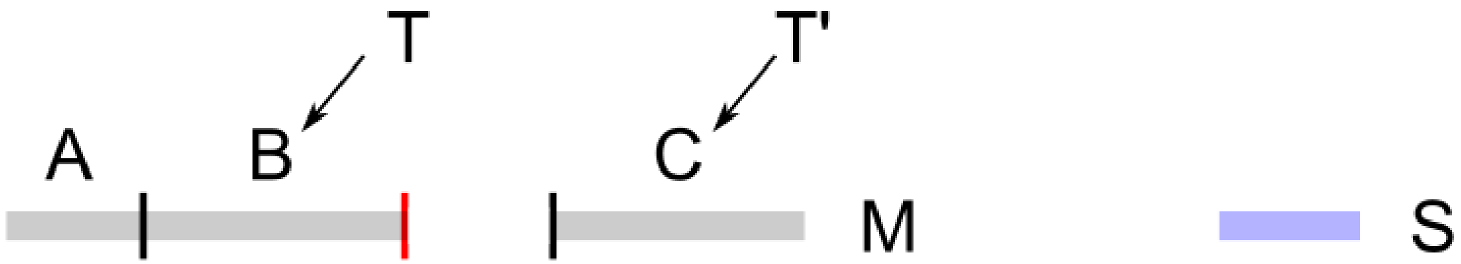

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## Phase 2

# Selfridge-Conway Procedure

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Let T = owner of the trimmed piece (T = B or C); let T' = B∪C \ T.


## Phase 2

4. Agent T' divides the trimmings S into three equal pieces (as per $v_{T'}$).

# Selfridge-Conway Procedure

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1. Agent A divides the cake into three equal pieces (as per $v_A$).
2. Agent B trims its favorite piece to create a two-way with its second-favorite.
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3. Agent C, then B, then A, in that order, pick a piece each from main cake M.
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Let T = owner of the trimmed piece (T = B or C); let T' = B∪C \ T.


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3. Agent C, then B, then A, in that order, pick a piece each from main cake M.
   - Agent B must pick the trimmed piece if agent C does not.

Let T = owner of the trimmed piece (T = B or C); let T' = B∪C \ T.


## Phase 2

4. Agent T' divides the trimmings S into three equal pieces (as per $v_{T'}$).
5. Agent T, then A, then T', in that order, pick a piece each from trimmings S.

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2. Agent B trims its favorite piece to create a two-way with its second-favorite.
   - Trimmings = S; Main cake M; Original cake = M∪S
3. Agent C, then B, then A, in that order, pick a piece each from main cake M.
   - Agent B must pick the trimmed piece if agent C does not.

Let T = owner of the trimmed piece (T = B or C); let T' = B∪C \ T.


## Phase 2

4. Agent T' divides the trimmings S into three equal pieces (as per $v_{T'}$).
5. Agent T, then A, then T', in that order, pick a piece each from trimmings S.

# Selfridge-Conway Procedure

# Selfridge-Conway Procedure

- Is any part of the cake left unassigned in the final allocation?

# Selfridge-Conway Procedure

- Is any part of the cake left unassigned in the final allocation? No.

# Selfridge-Conway Procedure

- Is the final allocation envy-free from agent C's perspective?

# Selfridge-Conway Procedure

- Is the final allocation envy-free from agent C's perspective? Yes.

# Selfridge-Conway Procedure

- Is the final allocation envy-free from agent C's perspective? Yes.
    - Within the main cake M, C does not envy A or B because it chooses first.

# Selfridge-Conway Procedure

- Is the final allocation envy-free from agent C's perspective? Yes.
  - Within the main cake M, C does not envy A or B because it chooses first.
  - Within the trimmings S, C does not envy A or B because:

# Selfridge-Conway Procedure

- Is the final allocation envy-free from agent C's perspective? Yes.
  - Within the main cake M, C does not envy A or B because it chooses first.
  - Within the trimmings S, C does not envy A or B because:
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# Selfridge-Conway Procedure

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    - If C is T', then it divides S into three equal pieces.

# Selfridge-Conway Procedure

- Is the final allocation envy-free from agent C's perspective? Yes.
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  - Within the trimmings S, C does not envy A or B because:
    - If C is T, then it chooses first in S.
    - If C is T', then it divides S into three equal pieces.
  - By additivity across M∪S, C does not envy A or B w.r.t. the entire cake.

# Selfridge-Conway Procedure

- Is the final allocation envy-free from agent B's perspective?

# Selfridge-Conway Procedure

- Is the final allocation envy-free from agent B's perspective? Yes.

# Selfridge-Conway Procedure

- Is the final allocation envy-free from agent B's perspective? Yes.
  - Within the main cake M, B does not envy A or C because of two-way tie.

# Selfridge-Conway Procedure

- Is the final allocation envy-free from agent B's perspective? Yes.
  - Within the main cake M, B does not envy A or C because of two-way tie.
  - Within the trimmings S, B does not envy A or C because:

# Selfridge-Conway Procedure

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# Selfridge-Conway Procedure

- Is the final allocation envy-free from agent B's perspective? Yes.
  - Within the main cake M, B does not envy A or C because of two-way tie.
  - Within the trimmings S, B does not envy A or C because:
    - If B is T, then it chooses first in S.
    - If B is T', then it cuts S into three equal pieces.

# Selfridge-Conway Procedure

- Is the final allocation envy-free from agent B's perspective? Yes.
  - Within the main cake M, B does not envy A or C because of two-way tie.
  - Within the trimmings S, B does not envy A or C because:
    - If B is T, then it chooses first in S.
    - If B is T', then it cuts S into three equal pieces.
  - By additivity across M∪S, B does not envy A or C w.r.t. the entire cake.

# Selfridge-Conway Procedure

- Is the final allocation envy-free from agent A's perspective?

# Selfridge-Conway Procedure

- Is the final allocation envy-free from agent A's perspective? Yes.

# Selfridge-Conway Procedure

- Is the final allocation envy-free from agent A's perspective? Yes.
  - Within the main cake M, A does not envy B or C because it was the cutter and it never gets the trimmed piece.

# Selfridge-Conway Procedure

- Is the final allocation envy-free from agent A's perspective? Yes.
  - Within the main cake M, A does not envy B or C because it was the cutter and it never gets the trimmed piece.
  - Within the trimmings S, A does not envy:

# Selfridge-Conway Procedure

- Is the final allocation envy-free from agent A's perspective? Yes.
  - Within the main cake M, A does not envy B or C because it was the cutter and it never gets the trimmed piece.
  - Within the trimmings S, A does not envy:
    - T' because it picks before T' does.

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    - T because of "irrevocable advantange" from Phase 1.

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# The Story of Envy-freeness

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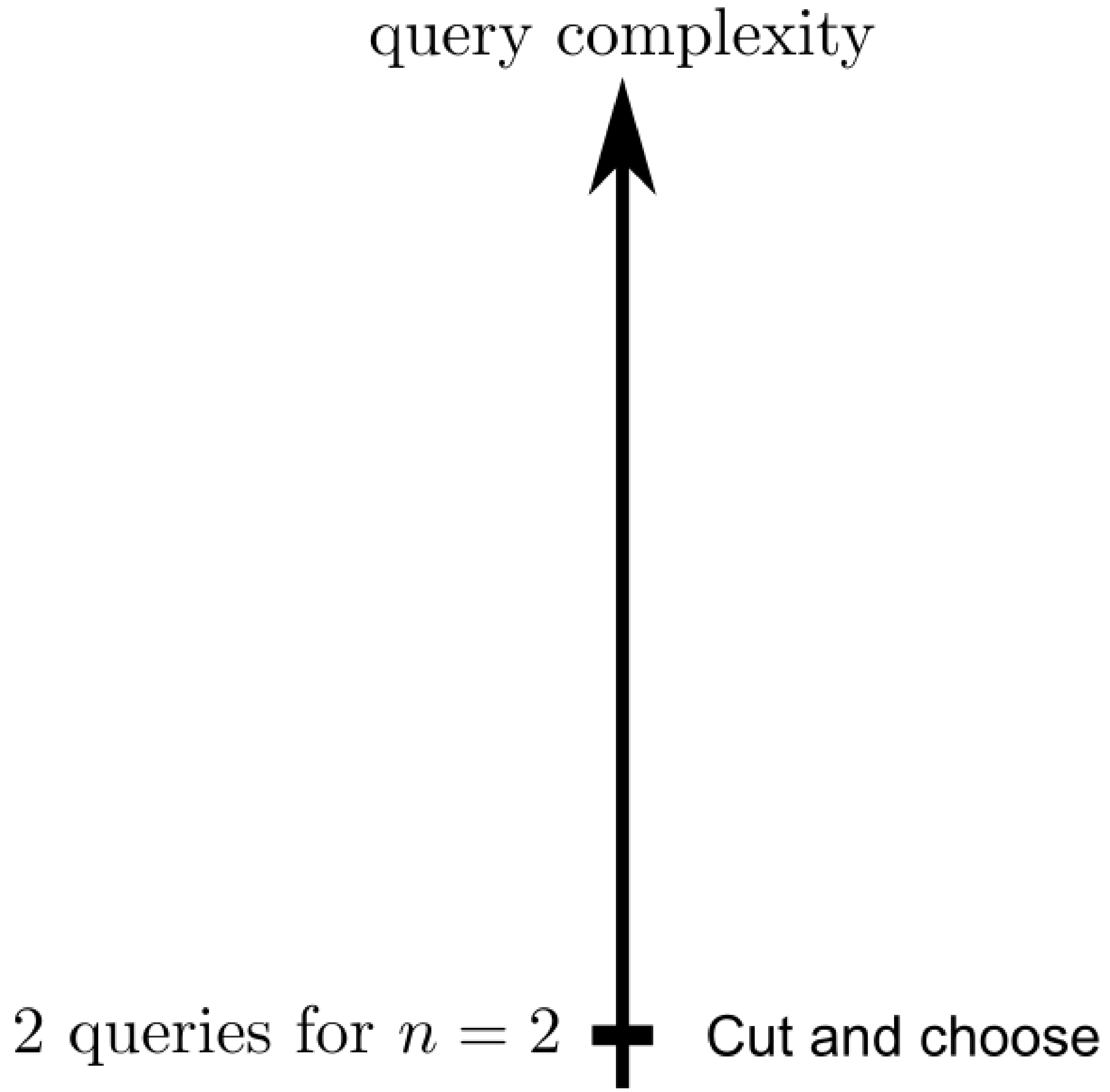

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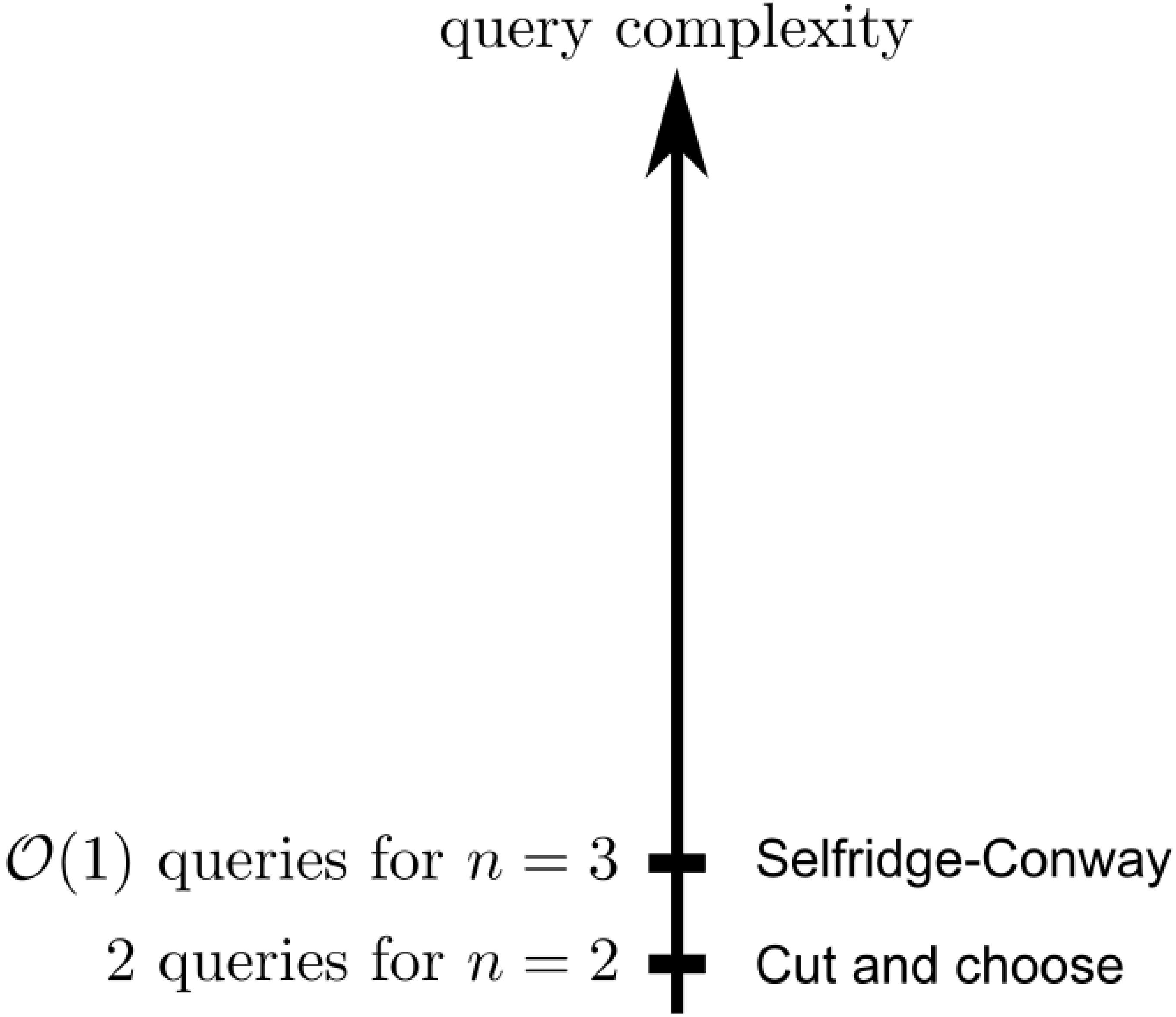


# The Story of Envy-freeness

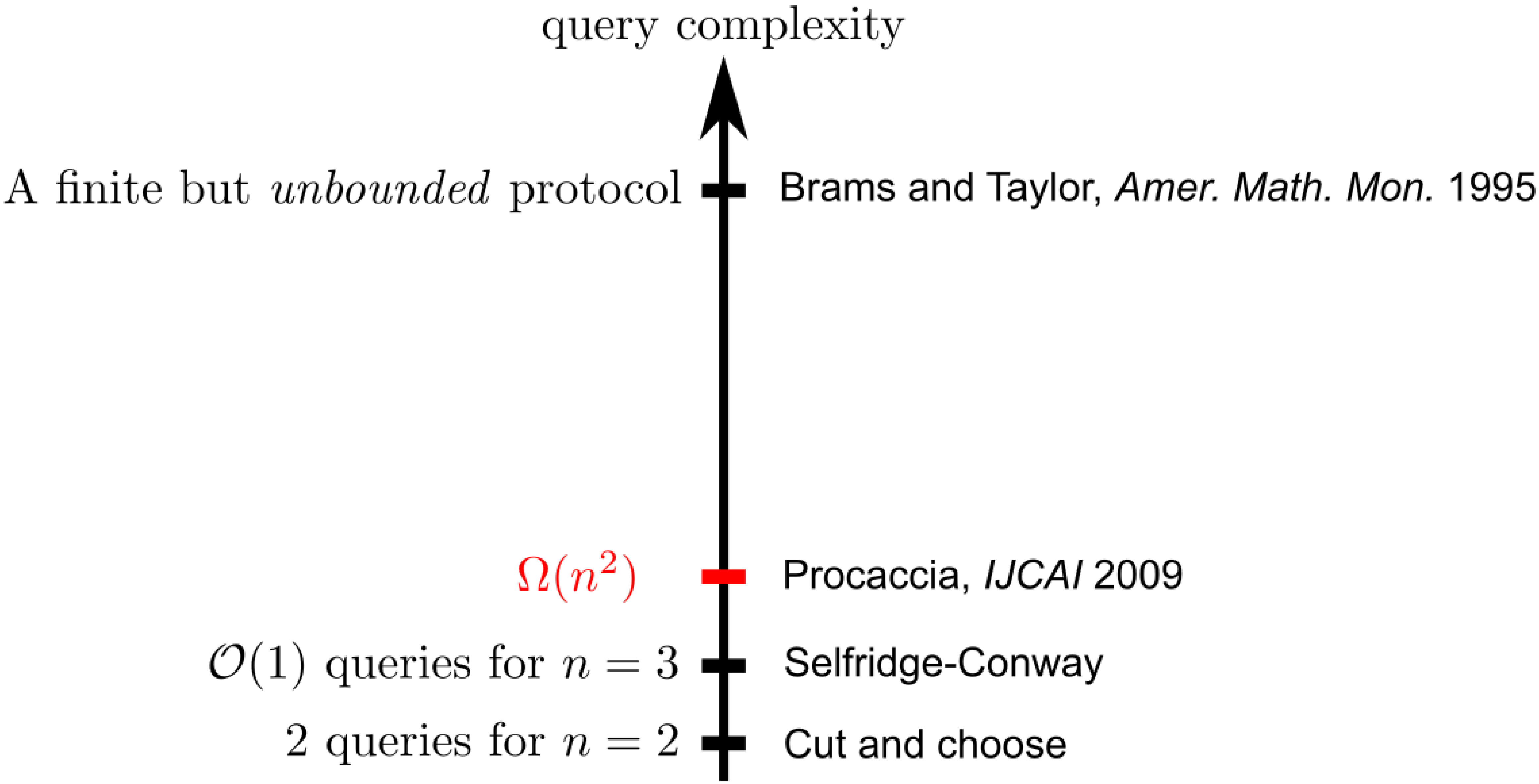
query complexity

A finite but *unbounded* protocol — Brams and Taylor, *Amer. Math. Mon.* 1995

$\mathcal{O}(1)$ queries for $n = 3$ — Selfridge-Conway

2 queries for $n = 2$ — Cut and choose

# The Story of Envy-freeness



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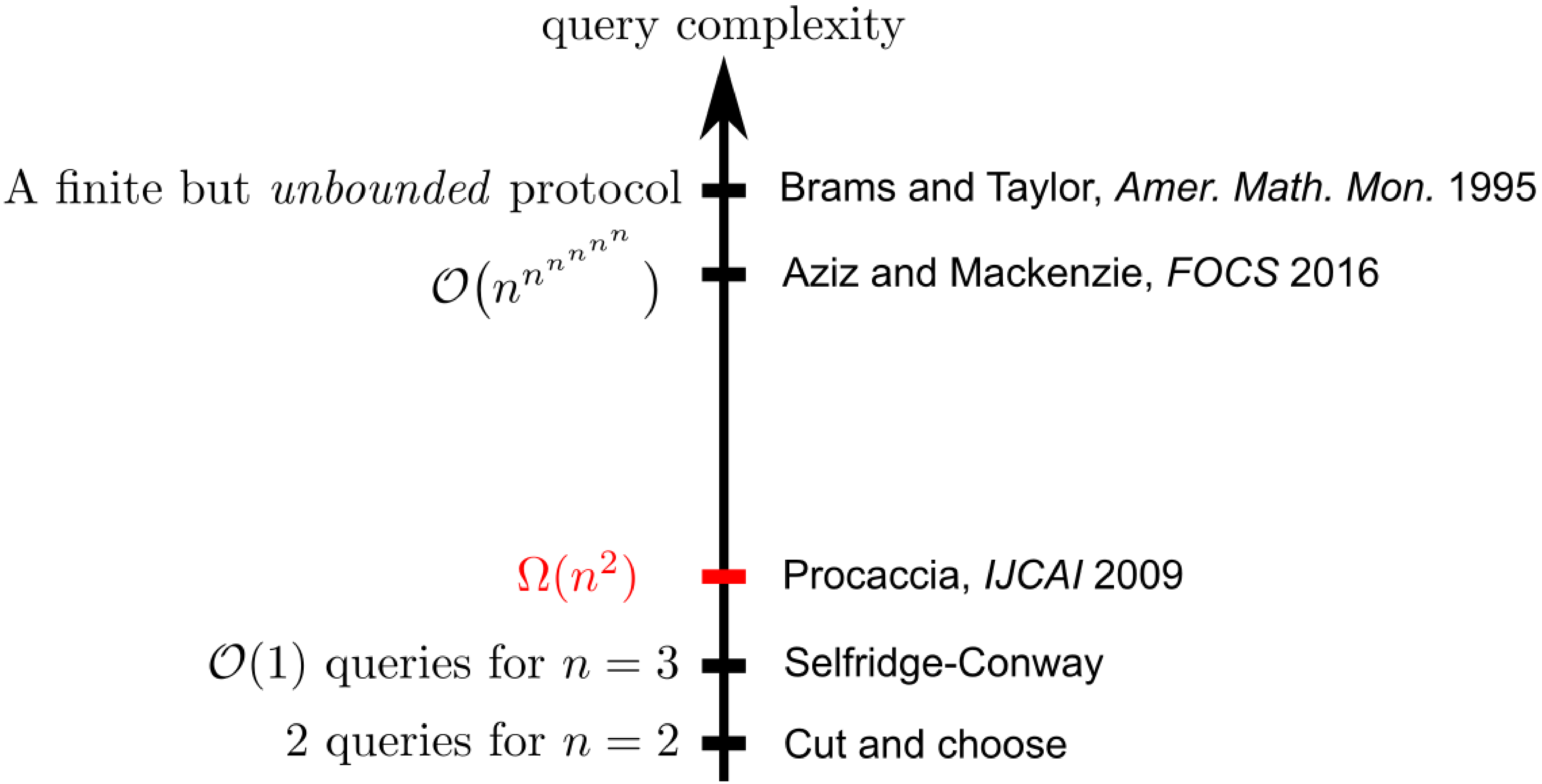

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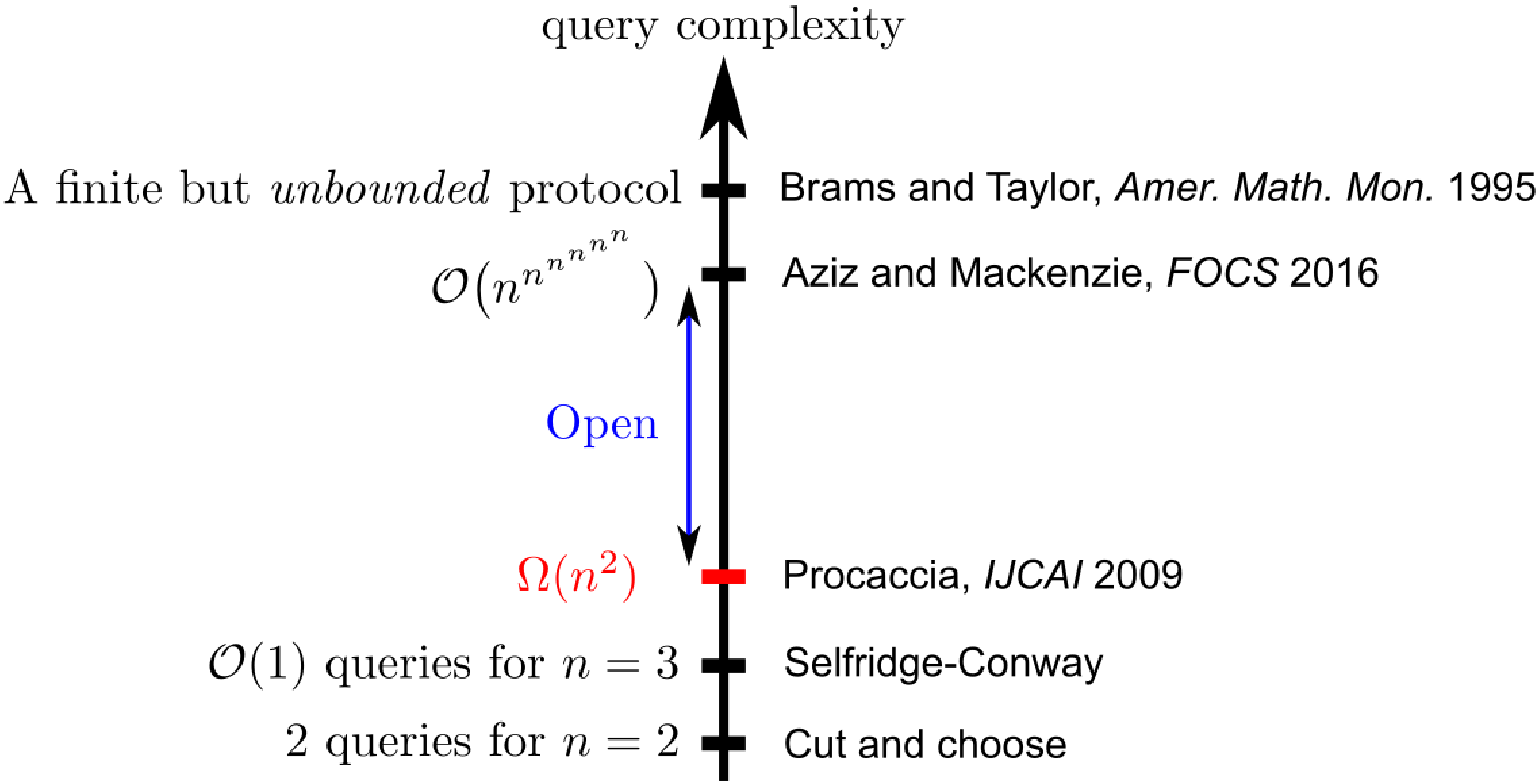

# Next Time

Fair Rent Division

# References

- Introduction to cake-cutting algorithms.

  Ariel Procaccia
  "*Cake Cutting Algorithms*"
  Chapter 13 in Handbook of Computational Social Choice

- Lecture by Ariel Procaccia on "Cake cutting" in the *Optimized Democracy* course.
  https://sites.google.com/view/optdemocracy/schedule